Journal of Statistical Physics, Vol. 21, No. 6, 1979

The Universal Metric Properties of Nonlinear Transformations

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Received May 29, 1979

The role of functional equations to describe the exact local structure of highly bifurcated attractors of $x_{n+1} = \lambda f(x_n)$ independent of a specific f is formally developed. A hierarchy of universal functions $g_r(x)$ exists, each descriptive of the same local structure but at levels of a cluster of 2^r points. The hierarchy obeys $g_{r-1}(x) = -\alpha g_r(g_r(x/\alpha))$, with $g = \lim_{r\to\infty} g_r$ existing and obeying $g(x) = -\alpha g(g(x/\alpha))$, an equation whose solution determines both g and α. For r asymptotic

$$g_r \sim g - \delta^{-r}h \qquad (*)$$

where $\delta > 1$ and h are determined as the associated eigenvalue and eigenvector of the operator $\mathscr{L}$:

$$\mathscr{L}[\psi] = -\alpha[\psi(g(x/\alpha)) + g'(g(x/\alpha))\psi(-x/\alpha)]$$

We conjecture that $\mathscr{L}$ possesses a unique eigenvalue in excess of 1, and show that this δ is the λ-convergence rate. The form (*) is then continued to all λ rather than just discrete λ_r and bifurcation values Λ_r and dynamics at such λ is determined. These results hold for the high bifurcations of any fundamental cycle. We proceed to analyze the approach to the asymptotic regime and show, granted $\mathscr{L}$'s spectral conjecture, the stability of the g_r limit of highly iterated λf's, thus establishing our theory in a local sense. We show in the course of this that highly iterated λf's are conjugate to g_r's, thereby providing some elementary approximation schemes for obtaining λ_r for a chosen f.

KEY WORDS: Recurrence; bifurcation; attractor; universal; functional equations; scaling; conjugacy; spectrum of linearized operator.

Work performed under the auspices of the U.S. Energy Research and Development Administration.

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0022-4715/79/1200-0669$03.00/0

1. INTRODUCTION

In a previous paper[1] (hereafter referred to as I), a viewpoint was advanced that detailed information about large stability sets of a recursion relation

$$x_{n+1} = \lambda f(x_n) \tag{1}$$

is available independent of the exact form of f for a wide class of functions. A heuristic argument (corroborated by computer computation) was offered to the effect that appropriate functional equations, free of reference to the recursion equation, furnish all this detailed quantitative information. Specifically, the exact distribution of points of large limit cycles of the recursion equation within local clusters is determined by a certain universal function $g^*(x)$ obeying a functional equation we conjectured to exist, but only approximately could specify. A parameter α implicated in that equation, and presumably determined by it collaterally with $g^*(x)$, plays the role of a fundamental scale factor: upon bifurcation of a high-order cycle, the points of the bifurcated cycle are identically distributed, save for a reduction in scale by the factor α. Another fundamental parameter δ, the convergence rate of a variety of universal details, was crudely determined from $g^*(x)$.

In the present paper we shall vindicate these conjectures in exhibiting an exact equation determining α and a universal function g closely related to g^* of I. Indeed, two functions $g^*(x)$ and $-\alpha g^*(g^*(x/\alpha))$ were discussed in I; these are the first two (g_1 and g_0, respectively) of an infinite sequence of functions $g_r(x)$ linked by the shift operation

$$g_{r-1}(x) = -\alpha g_r(g_r(x/\alpha)) \tag{2}$$

The equation

$$g(x) = -\alpha g(g(x/\alpha)) \tag{3}$$

is obeyed by $g(x) = \lim_{r\to\infty} g_r(x)$, a function determining the local distribution of infinite clusters of elements of *all* the infinite attractors of (1). We then proceed to determine $g_r(x)$ for large r in terms of an auxiliary function $h(x)$ obeying a functional equation implicating δ and determining both $h(x)$ and δ. Utilizing (2), one can then step down to determine from a g_r for $r \gg 1$ the g^* of I. Thus, as conjectured in I, the entire local structure of high-order stability sets of (1) is determined in a framework liberated from (1).

With the structure of the infinite limiting attractors laid out in Sections 2 and 3, we investigate in Section 4 the asymptotic approach to this structure. The equation obeyed by h is a linear functional eigenvalue equation, whose eigenvalues in excess of 1 lead to convergence of g_r to g; the eigenvalues bounded by 1 in absolute value represent potential instabilities.

However, in analyzing the large-n approach to a g_r, we discover that exactly these eigenvalues lead to convergence, so that in the infinite-n limit, g_r possesses no unstable components. In this fashion, though, the large eigenvalues destroy convergence to g_r. We discover, however, that the choice of the λ_r dictated by the recursion equation exactly suppresses this instability *providing* there is a *unique* eigenvalue in excess of 1. This eigenvalue is δ and we conjecture its uniqueness. Proof of this conjecture would constitute a local proof of universality. (At present we have only computer corroboration.)

Finally, in Section 5 we discuss techniques for the solution of the fundamental functional equations and various approximation schemes.

2. THE SEQUENCE $\{g_r\}$ OF UNIVERSAL FUNCTIONS AND THE BASIC FUNCTIONAL EQUATION

As heuristically argued in I, defining

$$g(\lambda, x) \equiv \lambda f(x)$$

where f possesses a differentiable zth-order maximum at $x = 0$,

$$f(0) - f(x) \propto |x|^z \qquad z > 1 \text{ for } |x| \text{ small}$$

we have

$$(-\alpha)^n g^{(2^n)}(\lambda_{n+1}, x/\alpha^n) \sim \mu g^*(x/\mu) \tag{4}$$

for large n, where $g^{(n)}(x)$ is the nth iterate of g:

$$g^{(2)}(x) \equiv g(g(x)); \qquad g^{(n+1)}(x) \equiv g(g^{(n)}(x))$$

and μ depends upon the specific form of f. Rescaling g^* on both height and width by μ removes all vestige of the specific form of f, and is accomplished through the definition of an absolute scale:

$$g^*(0) = 1$$

We understand this absolute rescaling implicitly, and simply write (4) as

$$(-\alpha)^n g^{(2^n)}(\lambda_{n+1}, x/\alpha^n) \sim g^*(x) \tag{5}$$

The value of λ, λ_{n+1} is determined by the condition

$$g^{(2^n)}(\lambda_n, 0) = 0, \qquad g^{(2^{n'})}(\lambda_n, 0) \neq 0 \qquad \text{for } n' < n$$

g^* accordingly describes a two-point cycle near $x = 0$, since $g^*(0) = 1$ and $g^*(1) = 0$ (see Fig. 1). With g^* the limit of (5) as $n \to \infty$, and universal for

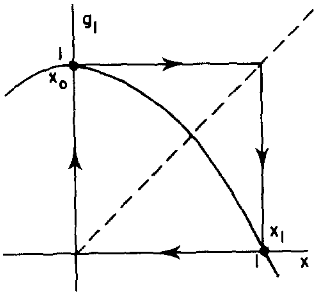

Fig. 1. The function g_1 normalized to $g_1(0) = 1$; the 2-cycle $x_0 \to x_1 \to x_0 \to x$, etc., is indicated.

all f of fixed z, $g^*(g^*(x))$ is itself universal, possessing 0 and 1 as fixed points. $g^*(g^*)$ is also a limit of highly iterated g's:

$$\begin{aligned} g^*(g^*(x)) &\sim (-\alpha)^n g^{(2^n)}(\lambda_{n+1}, g^{(2^n)}(\lambda_{n+1}, x/\alpha_n)) \\ &= (-\alpha)^n g^{(2^{n+1})}(\lambda_{n+1}, x/\alpha^n) \end{aligned}$$

or,

$$\begin{aligned} g_0 \equiv -\alpha g^*(g^*(x/\alpha)) &\sim (-\alpha)^{n+1} g^{(2^{n+1})}(\lambda_{n+1}, x/\alpha^{n+1}) \\ &\sim (-\alpha)^n g^{(2^n)}(\lambda_n, x/\alpha^n) \end{aligned} \tag{6}$$

Thus, $g^*(x)$ is obtained from $g_0(x)$ by increasing λ into the next bifurcation; conversely, g_0 describes a one-cycle near $x = 0$ (Fig. 2). Both g_0 and $g^* \equiv g_1$ describe the identical local structure of the elements of a large 2^n-cycle near $x = 0$, but at different "magnifications." Generalizing,

$$g_r(x) \equiv \lim_{n\to\infty} (-\alpha)^n g^{(2^n)}(\lambda_{n+r}, x/\alpha^n) \tag{7}$$

again describes the identical local structure, but now at a level of magnification such that 2^r elements of the cycle are clustered about the central bump (Fig. 3). From the definition (7),

$$g_{r-1}(x) = -\alpha g_r(g_r(x/\alpha)) \tag{8}$$

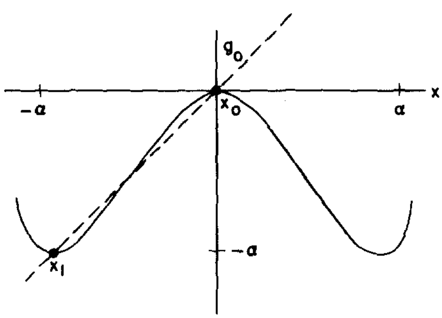

Fig. 2. The function g_0 corresponding to g_1 normalized as in Fig. 1; the locations of x_0 and x_1 are magnified by $-\alpha$.

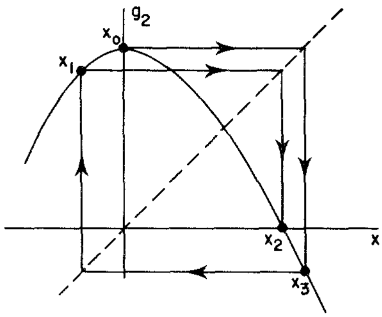

Fig. 3. The function g_2; the 4-cycle $x_0 \to x_3 \to x_1 \to x_2 \to x_0$, etc., is indicated. x_0 and x_1 are as in Figs. 1 and 2, but now *reduced* from Fig. 1 by $-\alpha$.

(Universality implies that all g_r are symmetric functions.) Given any g_r, simply iterating produces all other g_m for $m < n$, and each contains the identical information.

When r is so large that n can become large and yet much smaller than r, then

$$g^{(2^n)}(\lambda_{n+r+1}) \approx g^{(2^n)}(\lambda_{n+r})$$

since $\lambda_m \to \lambda_\infty$ and $\lambda_{n+r+1} \approx \lambda_{n+r}$: n must be of the order of r before any error in λ_{n+r} can become significant. Alternatively, the central bump suffers very slight distortion to accommodate the infinite attractor when it already accommodates a very large attractor. That is, we intuitively conjecture that the limit

$$\lim_{r\to\infty} g_r(x) \equiv g(x)$$

exists. This granted, (8) implies that g satisfies

$$g(x) = -\alpha g(g(x/\alpha)) \tag{9}$$

Qualitatively, $g(x)$ looks like the curve of Fig. 3. Yet, $g(x)$ contains different information from $g_r(x)$ for any finite r: quite simply, any two stability points located by g_r possess a minimum separation, which is not true for g. Rather, g represents a different level of universal distribution of stability points: the entirety of $g_1(x)$ is collapsed to a point at the level of g. (This is again a reflection of the Cantor set-like nature of highly bifurcated stability sets—indeed, infinitely bifurcated.)

An alternate definition of g in the n-limit sense is

$$g(x) = \lim_{n\to\infty} (-\alpha)^n g^{(2^n)}(\lambda_\infty, x/\alpha^n) \tag{10}$$

since λ_∞ is a finite, perfectly definite value of λ. It is clear from (10) why we succeeded in obtaining an exact equation for g: the λ-shifting which frustrated our attempt at an exact equation for g^* in I is here absent. There is, however, a strong price to be exacted for this grace: unlike the hoped-for equation in I, (9) must be recursively unstable. To understand this, let us rederive (9) from (10). Define

$$\tilde{g}_n(x) \equiv (-1)^n \beta_n g^{(2^n)}(\lambda_\infty, x/\beta_n) \tag{11}$$

or

$$(1/\beta_n)\tilde{g}_n(\beta_n x) = (-1)^n g^{(2^n)}(\lambda_\infty, x)$$

Then

$$\begin{aligned}(-1)^n g^{(2^{n+1})}(\lambda_\infty, x) &= (-1)^n g^{(2^n)}(\lambda_\infty, (-1)^n g^{(2^n)}(\lambda_\infty, x)) \\ &= \frac{1}{\beta_n}\, \tilde{g}_n((-1)^n \beta_n g^{(2^n)}(\lambda_\infty, x)) \\ &= \frac{1}{\beta_n}\, \tilde{g}_n(\tilde{g}_n(\beta_n x))\end{aligned}$$

or,

$$-\frac{1}{\beta_{n+1}}\, \tilde{g}_{n+1}(\beta_{n+1} x) = \frac{1}{\beta_n}\, \tilde{g}_n(\tilde{g}_n(\beta_n x))$$

or, with $\beta_{n+1}/\beta_n \equiv \alpha_n$,

$$\tilde{g}_{n+1}(x) = -\alpha_n \tilde{g}_n(\tilde{g}_n(x/\alpha_n)) \tag{12}$$

Setting an absolute scale

$$\tilde{g}_n(0) = 1 \qquad \text{for all } n$$

(12) implies that $g_n(x)$ determines α_n:

$$\tilde{g}_{n+1}(0) = -\alpha_n \tilde{g}_n(\tilde{g}_n(0))$$

or

$$1 = -\alpha_n \tilde{g}_n(1) \tag{13}$$

Accordingly, choosing a $\tilde{g}_0(x)$ satisfying $\tilde{g}_0(0) = 1$ and possessing a zth order maximum at $x = 0$, we can use (12) and (13) to recursively generate $\tilde{g}_n(x)$. Should $\tilde{g}_n(x) \to g(x)$, then

$$g(x) = -\alpha g(g(x/\alpha))$$

with $\alpha = \lim_{n\to\infty} \alpha_n$.

Apart from manipulations, the regimen of (12) and (13) is simply a machine to perform the (attempted) limit of (10) starting with a $\tilde{g}_0$ that is essentially $\lambda_\infty f(x)$, or more exactly

$$\tilde{g}_0(x) = f(\lambda_\infty x) \tag{14}$$

[So that $\tilde{g}_0(0) = 1$, we have rescaled on height and width by λ_∞: (9) is invariant to such rescaling.] Since λ_∞ depends upon f,

$$\tilde{g}_0(x) = f(\lambda_\infty(f)x) \tag{15}$$

Indeed, for any f of our class, $\tilde{g}_0$ as given by (15) must result in $\tilde{g}_n \to g$. *However*, for

$$\tilde{g}_0(x) = f(ax), \qquad a \neq \lambda_\infty$$

it must be *impossible* for $\tilde{g}_n$ to converge: unless $a = \lambda_\infty$ for some harmonic sequence, there is no infinite attractor and no sequence of g_r's converging to g. For example, if we choose

$$\tilde{g}_0(x) = 1 - ax^2$$

then unless a is chosen at special isolated values, the $\tilde{g}_n$ will not converge. Rather, a could in general be a value λ_m; after a number of iterations, $\tilde{g}_n$ would, by definition, be a g_r [Eq. (7)] approximately. Since (12) is (8) (with some rescaling), successive iterations would move toward g_0 rather than g and then divergently away. Figure 4 represents a suggestive picture of the situation. That is, the fixed point g is repellent, and unless $\tilde{g}_0$ is correctly chosen so that the $\tilde{g}_n$ will "aim" into g, they will at first approach $g(\lambda_r \simeq \lambda_\infty$ so far as $g^{(2^n)}$ is concerned until $n \sim r)$, but then diverge away from it along the "path" of decreasing g_r's. That is, (9) in general defines a recursively unstable problem.

This instability can, however, be turned to excellent advantage. Since an arbitrary $\tilde{g}_0$ will lead to divergence, a "good" $\tilde{g}_0$ must already be a good approximation to g. With $g(0) = 1$, (9) implies that

$$g(1) = -1/\alpha$$

By (14) one should then estimate

$$f(\lambda_\infty(f)) = \tilde{g}_0(1) \simeq -1/\alpha \tag{16}$$

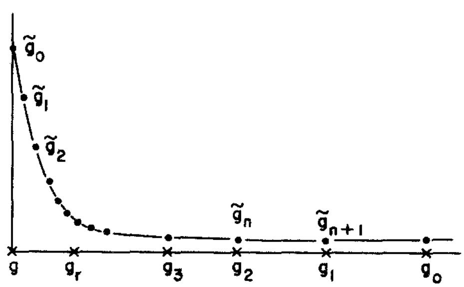

Fig. 4. The sequence of g_r's is indicated as points along the x axis. The iterates of $\tilde{g}_0$ are shown at first approaching g, the trajectory spending many iterations in the vicinity of g (or g_r for large r), but ultimately diverging away near low-lying g_r's.

that is, the instability of (9) provides an estimation of $\lambda_\infty(f)$ for any suitable f. The closer an f is to g (in some sense), the better the estimate. For example, consider

$$x_{n+1} = \lambda(1 - 2x_n^2)$$

or

$$f = 1 - 2x^2$$

and

$$f(\lambda_\infty x) = 1 - 2\lambda_\infty^2 x^2$$

With $\alpha = 2.5029\cdots$ for $z = 2$, we obtain from $1 - 2\lambda_\infty^2 \simeq -1/\alpha$ a value of $\lambda_\infty \simeq 0.8365$, to be compared to $\lambda_\infty = 0.8370$ for the limit of 2^n-cycles. In Section 5 we shall pursue this idea to obtain a technique for solving (9).

3. THE INFINITESIMAL λ SHIFT AND CONVERGENCE

Increasing λ from λ_n to λ_{n+1} maps g_r into g_{r+1} for all r. Calling this operation R, the λ shift, we write

$$R(g_r) = g_{r+1} \tag{17}$$

In I, R was applied to g_0 to produce g_1. Equation (8), written as

$$g_{r-1} = -\alpha g_r(g_r(x/\alpha)) \equiv L(g_r)$$

accomplishes the operation inverse to R. The combined operation

$$B \equiv L \cdot R$$

is the bifurcation transformation of I, which serves as an identity on the sequence $\{g_r\}$:

$$B(g_r) = g_r, \qquad r = 0, 1, \ldots \tag{18}$$

or, each g_r is a fixed point of the transformation B. We select g_1 by imposing the conditions

$$g_1(0) = 1, \qquad g_1(1) = 0 \tag{19}$$

and our universality conjecture is phrased in this language by saying the fixed point g_1 of B is *stable*, so that if any $\tilde{g}_0$ satisfying (19) with a zth-order maximum is chosen,

$$\tilde{g}_{n+1} = B[\tilde{g}_n]$$

will result in

$$\tilde{g}_n \to g_1$$

for the zth-order maximum universal g_1. The empirical computer evidence for universality, together with the instability of (12), means that R stabilizes B, as R reduces trivially to the identity only for the fixed-point g. Indeed, our approximate modelings of R in I resulted in recursively stable functional equations. We now determine R restricted to operation on g_r's in the limit of infinite r. The interchangeability of r and n in (7)

$$(-\alpha)^n g^{(2^n)}(\lambda_{n+r}, x/\alpha_n) \sim (-\alpha)^{n-s} g^{(2^{n-s})}(\lambda_{n+(r-s)}, x/\alpha^{n-s}), \qquad n - s \gg 1$$

together with the shifting operators implies that our study shall provide information about large-n convergence properties, and so determine δ as well.

We want to compute

$$\delta g^{(2^n)}(\lambda_{n+r}, x) \equiv g^{(2^n)}(\lambda_{n+r+1}, x) - g^{(2^n)}(\lambda_{n+r}, x) \tag{20}$$

Defining

$$\delta g_r(x) \equiv (-\alpha)^n \, \delta g^{(2^n)}(\lambda_{n+r}, x/\alpha^n) \tag{21}$$

(20) becomes

$$\delta g_r(x) = g_{r+1}(x) - g_r(x) = (R - 1)(g_r) \tag{22}$$

Substituting (22) in (8),

$$\begin{aligned} g_r(x) &= -\alpha(g_r + \delta g_r)[g_r(x/\alpha) + \delta g_r(x/\alpha)] \\ &= -\alpha g_r(g_r(x/\alpha)) - \alpha g_r'(g_r(x/\alpha))\, \delta g_r(x/\alpha) \\ &\quad - \alpha\, \delta g_r(g_r(x/\alpha)) + O((\delta g_r)^2) \\ &= g_{r-1}(x) - \alpha[\delta g_r(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\, \delta g_r(x/\alpha)] + O((\delta g_r)^2) \end{aligned} \tag{23}$$

or,

$$\delta g_{r-1}(x) = -\alpha[\delta g_r(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\, \delta g_r(x/\alpha)] + O((\delta g_r)^2) \tag{24}$$

Since $g_r \to g$, $\delta g_r \to 0$, and in the limit of infinite r

$$\begin{aligned} \delta g_{r-1}(x) &\sim -\alpha[\delta g_r(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\, \delta g_r(x/\alpha)] \\ &\sim -\alpha[\delta g_r(g(x/\alpha)) + g'(g(x/\alpha))\, \delta g_r(x/\alpha)] \end{aligned} \tag{25}$$

Separating $\delta g_r(x)$ as

$$\delta g_r(x) = \eta_r h(x) \tag{26}$$

with $\eta_r \to 0$ as $r \to \infty$, we obtained a closed equation for $h(x)$—the generator of infinitesimal λ shifts—and an equation for η_r:

$$\eta_{r-1} = \delta \eta_r \tag{27}$$

and

$$h(x) = -(\alpha/\delta)[h(g(x/\alpha)) + g'(g(x/\alpha))h(x/\alpha)] \tag{28}$$

In fact, (28) represents a rederivation of (28) of I, where (28) of I was an approximate realization of R applied to g_0, the approximation consisting of "mild" λ shifting, which becomes rigorous in the present context, and in this context, involving g and not g_1. Given $g(x)$ and α obeying (9), (28) determines both $h(x)$ and δ and defines a recursively stable equation. We return to this in the last section.

Equation (27) is trivially solved:

$$\eta_r = \delta^{-r} \tag{29}$$

so that

$$\delta g_r(x) = \delta^{-r} h(x) \tag{30}$$

or,

$$g_{r+1}(x) - g_r(x) = \delta^{-r} h(x), \qquad r \gg 1 \tag{31}$$

Summing (31) from $r = r_0$ to ∞, we obtain

$$g_{r_0}(x) = g(x) - \frac{\delta^{-r_0}}{1 - \delta^{-1}} h(x), \qquad r_0 \gg 1 \tag{32}$$

so that $g_r \to g$ (asymptotically) geometrically at the rate δ.

We now show that the δ determined by (28) and (9) is the δ of I: the argument is that of I resulting in (13) made exact. By (30),

$$\delta g_r(x) = \delta^{-1}\, \delta g_{r-1}(x)$$

which, by (21) reads for $x = 0$

$$(-\alpha)^n\, \delta g^{(2^n)}(\lambda_{n+r}, 0) = \delta^{-1}(-\alpha)^{n+1}\, \delta g^{(2^{n+1})}(\lambda_{n+r}, 0)$$

or

$$\delta g^{(2^{n-r})}(\lambda_n, 0) = -(\alpha/\delta)\, \delta g^{(2^{n-(r-1)})}(\lambda_n, 0)$$

or,

$$\delta g^{(2^s)}(\lambda_n, 0) = -(\alpha/\delta)\, \delta g^{(2^{s+1})}(\lambda_n, 0) \qquad \text{for } 1 \ll s \ll n$$

Thus, for n very large, the change in $g^{(2^r)}$ is the constant multiple $-\delta/\alpha$ of the change in $g^{(2^{r-1})}$ induced by increasing λ_n to λ_{n+1} for all r except for the very small (initial transient) and very large (the bottom of the g_r sequence). Accordingly,

$$(-1)^n\, \delta g^{(2^n)}(\lambda_n, 0) \sim (\delta/\alpha)^n\, \delta g^{(1)}(\lambda_n, 0) = (\lambda_{n+1} - \lambda_n)(\delta/\alpha)^n$$

in the sense of logarithms. Since

$$\begin{aligned}(-\alpha)^n\, \delta g^{(2^n)}(\lambda_n, 0) &= (-\alpha)^n g^{(2^n)}(\lambda_{n+1}, 0) - (-\alpha)^n g^{(2^n)}(\lambda_n, 0)\\ &= g_1(0) - g_0(0) \sim 1\end{aligned}$$

we have

$$\lambda_{n+1} - \lambda_n \sim \delta^{-n}$$

logarithmically, or $(\lambda_{n+1} - \lambda_n)/(\lambda_{n+2} - \lambda_{n+1}) \to \delta$ as $n \to \infty$, which is the original definition of δ in I.

Combining (9) and (28), we can obtain $g(x)$, $h(x)$, α, and δ. By (32) we next obtain g_r for large r, and then by repeated application of (8) obtain low-lying g_r's. We have thus succeeded in determining all local quantitative properties of all highly bifurcated (and infinite) attractors of (1) in a framework independent of (1), and its unspecified $f(x)$. *The theory of high-order attractors is fully posed in a functional equation framework*, and represents the common residue of all specifically posed recursion equations $x_{n+1} = \lambda f(x_n)$.

It is important to make two observations pertaining to Eq. (28) at this point: one concerning the uniqueness of δ and the other concerning the linearity of (28) to any scaling of h.

Equation (32) can be derived by setting

$$g_r(x) = g(x) + \eta_r(x)$$

substituting in (8), and expanding to first order in η: we are simply analyzing the manner of approach of g_r to g. The separation of (26), namely,

$$\eta_r(x) = \eta_r h(x) = \delta^{-r} h(x)$$

demonstrates that

$$\lim_{r \to \infty} g_r = g$$

provided the eigenvalue δ of (28) is strictly greater than 1. In fact, it is easy to see that $\delta = 1$ with $h(x) = g(x) - xg'(x)$ exactly satisfies (28). To see the significance of this solution, observe that

$$g(x) - xg'(x) = (1 - x\, d/dx)g(x)$$

is exactly the generator of infinitesimal magnifications:

$$\begin{aligned}(1 + \mu)g(x/1 + \mu) &= (1 + \mu)g(x - x\mu) + O(\mu^2)\\ &= g(x) + \mu(g(x) - xg'(x)) + O(\mu^2)\end{aligned}$$

However, the magnifications comprise a degeneracy group of (9): if $g(x)$ obeys (9), then so too does $\mu g(x/\mu) \equiv g_\mu(x)$:

$$g_\mu\left(g_\mu\left(\frac{x}{\alpha}\right)\right) = \mu g\left(\frac{1}{\mu}\,\mu g\left(\frac{x}{\alpha\mu}\right)\right) = \mu g\left(g\left(\frac{x/\mu}{\alpha}\right)\right) = -\frac{1}{\alpha}\,\mu g\left(\frac{x}{\mu}\right) = -\frac{1}{\alpha}\,g_\mu(x)$$

Thus, the r-independent piece of λ_r corresponding to the eigenvalue $+1$ simply represents a convergence of g_r to a suitably magnified g; by *choosing*

$g(0) = 1$, this freedom is eliminated, and the r-independent piece of η_r set to 0.

We shall see in the next section that a spectrum of eigenvalues bounded by 1 in absolute value also exists. Anticipating some of that discussion, it turns out that g_r is orthogonal to the span of this part of the spectrum, so that only the large (convergence-producing) eigenvalues matter here.

Observe that (28) is linear in h, so that if $h(x)$ is a solution, so too is $\mu h(x)$ for any μ. That is, $h(0)$, say, is free. By (32), with $g(0) = 1$ by convention, this leaves $g_r(0)$ free in the asymptotic-r regime. However, a definite choice of $h(0)$ is necessary to ensure that $g_0(0) = 0$. A different choice of $h(0)$ would, for no number of iterations of (8), result in a g_r satisfying $g_r(0) = 0$.

It is easy to comprehend the meaning of other choices of $h(0)$. Since

$$g_r(x) = g(x) - \delta^{-r}h(x)$$

if $\bar{h}(x)$ [for a definite $\bar{h}(0)$] guarantees that $g_0(0) = 0$, then

$$\bar{h}_1(x) \equiv \delta\bar{h}(x)$$

produces a $g_1(x)$ such that $g_1(0) = 0$: that is, by increasing $h(0)$ we need perform fewer iterations to obtain a g_r satisfying $g_r(0) = 0$. Differently put, the absolute size of $h(x)$ is logarithmically periodic with period $\log \delta$: if

$$\bar{h}(x) \Rightarrow g_0(0) = 0$$

then

$$\delta^n\bar{h}(x) \Rightarrow g_n(0) = 0$$

All this means is that as $\log \bar{h}(0)$ is increased by $\log \delta$, one has moved through an entire bifurcation. Choices of $h(0) \neq \delta^n\bar{h}(0)$ determine a sequence of g_r's whose λ's are chosen *not* at λ_n's, but rather at intermediate values of λ between λ_n and λ_{n+1}. In particular, there is choice of $h(0) \equiv H(0)$ such that the λ's are the *bifurcation* values Λ_n. That is, our results determine the behavior of stability points not just at those values of λ_n such that $\bar{x}$ is an element, but indeed the entire behavior as λ is *continually* increased to λ_∞. The reason is simple: since $\lambda_n \sim \lambda_\infty - \delta^{-n}$, $\delta^{-n} \sim \lambda_\infty - \lambda_n$, and $g_r(x)$ is

$$g_r(x) = g(x) - (\lambda_\infty - \lambda_n)h(x)$$

and so,

$$g_\lambda(x) \equiv g(x) - (\lambda_\infty - \lambda)h(x)$$

is the continuation from discrete λ to continuous λ. Deviations of λ from

λ_∞ are most naturally measured logarithmically to the base δ: the bifurcation values Λ_n obey

$$\lambda_\infty - \Lambda_n \sim \delta^{-n}$$

as do the λ widths of a given harmonic

$$\Lambda_{n+1} - \Lambda_n \sim \delta^{-n}$$

A given "kind" of cycle recurs in the next harmonic periodic logarithmically. Moreover, had we started with, say, a stable three-cycle, bifurcated to the six-cycle, and considered $\lambda_n \equiv \lambda$ such that the 3×2^n cycle is stable and includes $\bar{x}$, then $g^{(3)}(\lambda, x)$ about $\bar{x}$ describes a two-cycle and

$$g_r(x) = (-\alpha)^n g^{(3\times 2^n)}(\lambda_{n+r}, x/(-\alpha)^n)$$

more generally are of the same character as the g_r's obtained from the harmonics of the two-cycle. Clearly (8) is again obeyed, leading to (9). With α and g unique solutions to (9) for a fixed z, we now realize that the entirety of the above treatment carries over *un*changed in every way to the structure of *every* highly bifurcated cycle of (1) no matter from which fundamental the bifurcations are obtained. That is, the local description of stability points at both the isolated-point and infinite-cluster level as well as α and δ are unique for every highly bifurcated cycle of (1) independent of f for any fixed z. Thus, in the so-called "chaotic" regime of (1) where most values of λ correspond to high bifurcation of a high-order fundamental, the local description of the attractor is essentially that of the g's.

We mention in passing that once g_r has been continued to a continuous index, Eq. (8) in the form

$$g_{r-s}(x) = (-\alpha)^s g_r^{(2^s)}(x/\alpha^s)$$

defines the notion of a continuous interaction, since every ingredient of the equation has received a natural continuation, save for the 2^s iterations.

4. THE APPROACH TO THE $\{g_r\}$ FIXED POINT

At this point we return to the fundamental question of the large-n limit of (7). In general f is not symmetric about $\bar{x}$ (although universality implies that even for an asymmetric f the g_r's must be symmetric) and the correct form of (7) is

$$g_r(x) = \lim(-\alpha)^n g^{(2^n)}(\lambda_{n+r}, x/(-\alpha)^n) \tag{33}$$

Defining

$$g_{r,n}(x) \equiv (-\alpha)^n g^{(2^n)}(\lambda_{n+r}, x/(-\alpha)^n) \tag{34}$$

we can immediately verify that

$$g_{r-1,n+1}(x) = -\alpha g_{r,n}(g_{r,n}(-x/\alpha)) \tag{35}$$

Viewing (8) as a fixed point (in n) of (35), we could establish our theory in at least a local sense by ascertaining the stability of (8). Thus, we are led to consider

$$g_{r,n}(x) = g_r(x) + \eta_{r,n}(x) \tag{36}$$

and attempt to show that

$$\lim_{n\to\infty} \eta_{r,n}(x) = 0$$

Substituting (36) in (35), with (8) valid at the fixed point, we have in linear approximation (in η)

$$\eta_{r-1,n+1}(x) = -\alpha[\eta_{r,n}(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\eta_{r,n}(-x/\alpha)] \tag{37}$$

[Neglecting the n index, and replacing $g_r \to g$, we find that (37) reduces to (25): we shall have more to say about the eigenvalues of (28).] Since the $g_r(x)$ have been normalized to $g(0) = 1$, in general

$$\lim_{n\to\infty} g_{r,n} \neq g_r$$

but rather $g_{r,n}$ will approach a suitably magnified g_r. [We could alternatively have defined $\eta_{r,n} = g_{r,n} - \mu_{r+n}g_r(x/\mu_{r+n})$ with $\lim_{t\to\infty}\mu_t = \mu \neq 1$.] We shall account for this with $\eta^*_{r,n} \sim \mu(g_r - xg_r')$ a piece of $\eta_{r,n}$ to be determined from (37). Also, by defining

$$\eta_{r,n} \equiv \psi_{r,n+r}; \qquad n + r \equiv t$$

we find that (37) becomes

$$\psi_{r-1,t}(x) = -\alpha[\psi_{r,t}(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\psi_{r,t}(-x/\alpha)]$$

so that (37) itself is insufficient to determine any $n + r$ dependence; rather,

$$\eta_{r,0} = g_{r,0} - g_r = \lambda_r f - g_r \tag{38}$$

shall serve as initial data to fix $\eta_{r,n}$ uniquely.

We proceed to solve (37) by an artificial quadrature. Setting

$$\eta_{r,n} \equiv \eta^1_{r,n} + \eta^2_{r,n} \tag{39}$$

we find that (37) becomes

$$\begin{aligned}\eta^1_{r-1,n+1}(x) &+ \alpha[\eta^2_{r,n}(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\eta^1_{r,n}(-x/\alpha) + \eta^2_{r,n}(g_r(x/\alpha))] \\ &= -\{\eta^2_{r-1,n+1}(x) + \alpha g_r'(g_r(x/\alpha))\eta^2_{r,n}(-x/\alpha)\} \equiv 0\end{aligned}$$

thereby defining η^2. That is (38) is replaced with the pair of equations

$$\eta^1_{r-1,n+1}(x) = -\alpha[\eta^1_{r,n}(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\eta^1_{r,n}(-x/\alpha) + \eta^2_{r,n}(g_r(x/\alpha))] \tag{40}$$

and

$$\eta^2_{r-1,n+1}(x) = -\alpha g_r'(g_r(x/\alpha))\eta^2_{r,n}(-x/\alpha) \tag{41}$$

However, by (8)

$$g'_{r-1}(x) = -g_r'(g_r(x/\alpha))g_r'(x/\alpha)$$

so that (41) becomes

$$\frac{\eta^2_{r-1,n+1}(x)}{g'_{r-1}(x)} = \alpha\,\frac{\eta^2_{r,n}(-x/\alpha)}{g_r'(x/\alpha)}$$

Since $g_r(x)$ is symmetric,

$$g_r'(-x/\alpha) = -g_r'(x/\alpha)$$

so that

$$\frac{\eta^2_{r-1,n+1}(x)}{g'_{r-1}(x)} = -\alpha\,\frac{\eta^2_{r,n}(-x/\alpha)}{g_r'(-x/\alpha)} \tag{42}$$

Defining

$$f_{r,n}(x) \equiv \frac{\eta^2_{r,n}(x)}{g_r'(x)} \tag{43}$$

we find that (39) reads

$$f_{r-1,n+1}(x) = -\alpha f_{r,n}(-x/\alpha)$$

with solution

$$f_{r,n}(x) = (-\alpha)^n f_{r+n,0}(x/(-\alpha)^n) \equiv (-\alpha)^n F_{r+n}(x/(-\alpha)^n)$$

so that we have for η^2

$$\eta^2_{r,n}(x) = (-\alpha)^n F_{r+n}(x/(-\alpha)^n)g_r'(x) \tag{44}$$

With (44), (40) now reads

$$\begin{aligned}\eta^1_{r-1,n+1} = {} & -\alpha\{\eta^1_{r,n}(g_r'(x/\alpha)) + g_r(g_r'(x/\alpha)) \\ & \times [\eta^1_{r,n}(-x/\alpha) + (-\alpha)^n F_{r+n}(g_r(x/\alpha)/(-\alpha)^n)]\}\end{aligned} \tag{45}$$

Setting

$$\eta^1_{r,n}(x) \equiv \eta^0_{r,n}(x) - (-\alpha)^n F_{r+n}(g_r(x)/(-\alpha)^n)$$

we can immediately verify from (45) that

$$\eta^0_{r-1,n+1}(x) = -\alpha[\eta^0_{r,n}(g_r(x/\alpha)) + g_r(g_r'(x/\alpha))\eta^0_{r,n}(-x/\alpha)]$$

that is, if $\eta^0_{r,n}$ obeys (37), then for any F_{r+n}, so too does

$$\eta_{r,n}(x) = \eta^0_{r,n}(x) + [(-\alpha)^n F_{r+n}(x/(-\alpha)^n)g_r'(x) - (-\alpha)^n F_{r+n}(g_r(x)/(-\alpha)^n)] \tag{46}$$

so that we have obtained a particular solution of (37). We now regard $\eta^0_{r,n}(x)$ to be a homogeneous transient to the F solution which we utilize to meet initial data. Specifically, we absorb all antisymmetric parts of $f(x)$ into F, leaving $\eta^0_{r,0}(x)$ and hence all $\eta^0_{r,n}(x)$ symmetric, and so obeying the "intrinsic" equation

$$\eta^0_{r-1,n+1}(x) = -\alpha[\eta^0_{r,n}(g_r(x/\alpha)) + g_r'(g_r(x/\alpha))\eta^0_{r,n}(x/\alpha)] \tag{47}$$

η^0 is viewed as built exclusively of g_r's with minimal dependence on the initial $f(x)$. [It will be seen that the decomposition of (39)–(41) is determined and not at all artificial if η^1 and η^2 are respectively taken to be the symmetric and antisymmetric parts of η.]

We now utilize the initial data (38):

$$\lambda_r f(x) - g_r(x) = \eta_{r,0}(x) = \eta^0_{r,0}(x) + F_r(x)g_r'(x) - F_r(g_r(x)) \tag{48}$$

With an overbar denoting symmetry and a circumflex denoting antisymmetry, (48) reads

$$\lambda_r \hat{f}(x) = \bar{F}_r(x)g_r'(x) \tag{49}$$

and

$$\lambda_r \bar{f}(x) - g_r(x) = \eta^0_{r,0}(x) + \hat{F}_r(x)g_r'(x) - \hat{F}_r(g_r(x)) - \bar{F}_r(g_r(x)) \tag{50}$$

By (49), $\bar{F}_r(x)$ is nonvanishing only when f is asymmetric, in which case it can absorb all the antisymmetry.

Let us specialize to the case $\hat{f} \neq 0$. We then have from (49)

$$\bar{F}_r(x) = \frac{\lambda_r}{g_r'(x)}\hat{f}(x)$$

and so, by (46), a piece of $\eta_{r,n}$ of the form

$$\eta^*_{r,n}(x) = \lambda_{r+n}\left\{(-\alpha)^n \frac{\hat{f}(x/(-\alpha)^n)}{g'(x/(-\alpha)^n)} g_r'(x) - (-\alpha)^n \frac{\hat{f}(g_r(x)/(-\alpha)^n)}{g'_{r+n}(g_r(x)/(-\alpha)^n)}\right\}$$

$$\underset{n\to\infty}{\sim} \lambda_\infty\left\{(-\alpha)^n \frac{\hat{f}(x/(-\alpha)^n)}{g'(x/(-\alpha)^n)} g_r'(x) - (-\alpha)^n \frac{\hat{f}(g_r(x)/(-\alpha)_n)}{g'(g_r(x)/(-\alpha)^n)}\right\}$$

With $1 - \bar{f}(x) \propto |x|^z$, $\hat{f}(x) \propto |x|^{z+\epsilon} \operatorname{sgn} x$ ($\epsilon > 0$: otherwise f not extreme at $x = 0$), and $g'(x) \propto |x|^{z-1} \operatorname{sgn} x$,

$$(\hat{f}/g')(x) \propto |x|^{1+\epsilon}$$

and so

$$\eta^*_{r,n}(x) \underset{n\to\infty}{\simeq} (-1)^n\alpha^{-n\epsilon}(|x|^{1+\epsilon}g_r'(x) - |g_r(x)|^{1+\epsilon}) \to 0$$

Thus, the g_r fixed point is stable against antisymmetric perturbations. As an example, if

$$f(x) = 1 - ax^2 - bx^3$$

then $\epsilon = 1$ and $\eta_{r,n}$ converge to zero geometrically at the rate $-\alpha^{-1}$, in perfect agreement with the computer data for this f, since the $\eta_{r,n}$ convergence rate is exactly the α_n convergence rate:

$$\alpha_n^{(r)} \equiv -\frac{g^{(2^n)}(\lambda_{n+r,0})}{g^{(2^{n+1})}(\lambda_{n+1,r+0})} = \alpha \frac{g_{r,n}(0)}{g_{r,n+1}(0)}$$

$$\approx \alpha\left\{1 + \frac{1}{g_r(0)}[\eta_{r,n}(0) - \eta_{r,n+1}(0)]\right\}$$

Accordingly, we consider now only symmetric f's so that

$$F_r(x) = \hat{F}_r(x)$$

and

$$\lambda_r f(x) - g_r(x) = \eta_{r,0}^0(x) + \hat{F}_r(x) g_r{}'(x) - \hat{F}_r(g_r(x)) \tag{51}$$

As a first observation, following the parenthetic remark below Eq. (37),

$$\eta_r(x) \equiv \hat{F}_r(x) g'(x) - \hat{F}_r(g(x))$$

with

$$\hat{F}_r(x) = \alpha^{-r} \hat{F}_0(\alpha^r x)$$

must obey

$$\eta_{r-1}(x) = -\alpha[\eta_r(g(x/\alpha)) + g'(g(x/\alpha))\eta_r(x/\alpha)]$$

For monomials,

$$\hat{F}_0(x) = |x|^z \operatorname{sgn} x, \qquad \hat{F}_{r-1}(x) = \alpha^{-z+1} \hat{F}_r(x)$$

and so

$$-\alpha[h^{(z)}(g(x/\alpha)) + g'(g(x/\alpha))h^{(z)}(x/\alpha)] = \alpha^{-z+1} h^{(z)}(x)$$

with

$$h^{(z)}(x) = |g(x)|^z \operatorname{sgn}(g(x)) - g'(x)|x|^z \operatorname{sgn} x$$

That is, the eigenvalue of (28) can assume any positive value less than or equal to 1 in addition to the value $\delta > 1$. These eigenvalues represent potential instabilities of the convergence of g_r to g. However, they are unexcited in every g_r exactly because they provide stable convergence of $g_{r,n}$ to g_r: The g_r meet no initial conditions save for their convergence to g. The potentially hazardous parts $h^{(z)}$ of a g_r are all shed in the approach of $g_{r,n}$ (for any suitable f) to g_r. That is,

$$g = \lim_{r\to\infty} g_r$$

must exist since all unstable eigenvalues are exactly those that vanish in the formation of g_r from $g_{r,n}$.

We return to (51) and now ask whether the $\hat{F}$'s can span the initial data, in which case $\eta^0 \equiv 0$ and our theory is complete. In fact, an $\eta_{r,n}$ built wholly from $\hat{F}$ must produce convergence at a rate $(-\alpha)^{1-z}$ for

$$\hat{F}_r(x) = \mu_r x + a_r x^z + \cdots \text{ higher order}$$

For example, if $f = 1 - zx^2 + \cdots$, the smallest value of z would be 3, providing a convergence rate α^{-2}. However, $\alpha_n^{(1)} \to \alpha$ at the rate δ^{-1} in this case. Since $\delta < \alpha^2$, only the δ rate survives asymptotically, and leaves the question as to how it enters $\eta_{r,n}$. This suggests that η^0 might not vanish in general, and so we examine the situation more carefully.

Returning to (35), define

$$r + n \equiv t, \qquad g_{r,n} \equiv \psi_{r+n,n} = \psi_{t,n}$$

so that

$$\psi_{t,n+1}(x) = -\alpha\psi_{t,n}(\psi_{t,n}(-x/\alpha)) \tag{52}$$

We next expand ψ about g:

$$\psi_{t,n} \equiv g + \omega_{t,n} \tag{53}$$

so that

$$\omega_{t,n+1}(x) = -\alpha[\omega_{t,n}(g(x/\alpha)) + g'(g(x/\alpha))\omega_{t,n}(-x/\alpha)] \equiv \mathscr{L}[\omega_{t,n}] \tag{54}$$

in linear approximation. Equation (54) is our familiar shift equation written in the form of (12). We already know a variety of eigenvalues of $\mathscr{L}$:

$$\psi_\rho(x) \equiv g^\rho(x) - x^\rho g'(x) \Rightarrow \mathscr{L}[\psi_\rho] = (-\alpha)^{1-\rho}\psi_\rho$$

and

$$h(x)\colon \quad \mathscr{L}[h] = \delta h$$

Expanding $\omega_{t,n}$ along these eigenvectors, we have

$$\omega_{t,n} = \sum_\rho c_t^\rho(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g') + c_t\,\delta^n h + \omega_{t,n}^0 \tag{55}$$

Observe at this point a strong similarity to (46), and yet with the difference that (55) possesses an isolated h piece plus

$$\sum_\rho c_t^\rho(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g') = (-\alpha)^n F_t(g/(-\alpha)^n) - (-\alpha)^n F_t(x/(-\alpha)^n)g'$$

This is the same form of particular solution as in (46) but constructed from g rather than g_r. Now, if $c_t\delta^n = c_{r+n}\delta^n \sim \delta^{-r}$, then the piece $c_t\delta^n h$ is a fixed (in n) perturbation about g. Now, $\mathscr{L}$ accounts for only first-order perturbative effects. One would obtain a second-order correction by expanding about $g + c_t\delta^n h$. However, this is the form of expansion about g_r that would modify the $\psi_\rho(x)$ to be

$$\psi_\rho \to g_r{}^\rho - x^\rho g_r{}' = (g^\rho - x^\rho g') - \delta^{-r}(\rho g^{\rho-1}h - x^\rho h')$$

Thus, assuming $c_t\delta^n \sim \delta^{-r}$, the correct form of (55) including all first-order $c_t{}^\rho$ dependence is

$$\begin{aligned}\omega_{t,n} = &\sum_\rho c_t{}^\rho(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g') \\ &+ c_t\delta^n\left(h + \sum_\rho c_t{}^\rho(-\alpha)^{n(1-\rho)}(\rho g^{\rho-1}h - x^\rho h')\right) + \omega^0_{t,n} \end{aligned} \tag{56}$$

which possesses extra h dependence beyond the $F_t(g_r)$ terms than does (46): even with $\omega^0_{t,n} = 0$, (56) has already included a transient piece $\eta^{0'}_{r,n}$, which is still constructed from the span of $h \oplus \{\psi\}$. We now pose the (strong) conjecture that $\omega^0_{t,n} = 0$:

Conjecture. The spectrum of the operator $\mathscr{L}$ is δ and $(-\alpha)^{1-\rho}$, $\rho \leqslant 1$, and, moreover, the spectrum is complete.

(We possess computational evidence for this conjecture at least for $z = 2$ and 4, which we discuss in Section 5.) Accordingly, we have

$$\begin{aligned}\omega_{t,n} = &\sum_\rho c_t{}^\rho(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g') + c_t\delta^n\left(h + \sum_\rho c_t{}^\rho(-\alpha)^{n(1-\rho)}\right. \\ &\left.\times\, (\rho g^{\rho-1}h - x^\rho h')\right)\end{aligned} \tag{57}$$

or

$$\begin{aligned}g_{r,n} = g &+ \sum_\rho c^\rho_{t+n}(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g') \\ &+ c_{r+n}\delta^n\left(h + \sum_\rho c^\rho_{r+n}(-\alpha)^{n(1-\rho)}(\rho g^{\rho-1}h - x^\rho h')\right)\end{aligned} \tag{58}$$

in linear approximation.

It is easy to extend (58) to an exact solution of (35), since the first-order terms are exactly the generators of conjugacy transformations connected to the identity. Defining

$$F_t(x) = \sum_\rho c_t{}^\rho x^\rho$$

we find that (58) becomes

$$\begin{aligned}g_{r,n} = &(g + c_{r+n}\delta^n h) + (-\alpha)^n F_{r+n}((-\alpha)^{-n}(g + c_{r+n}\delta^n h)) \\ &- (-\alpha)^n(g + c_{r+n}\delta^n h)' F_{r+n}(x/(-\alpha)^n)\end{aligned} \tag{59}$$

Defining

$$S_{t,n}(x) = x + (-\alpha)^n F_t(x/(-\alpha)^n)$$

we have that (59) constitutes the leading approximation to

$$g_{r,n} = S_{r+n,n} \circ (g + c_{r+n}\delta^n h) \circ S_{r+n,n}^{-1} \tag{60}$$

However, defining $c_t = \delta^{-t} d_t$, we have

$$g + c_{r+n}\delta^n h = g + d_{r+n}\delta^{-r} h \equiv \tilde{g}_{r,r+n} + O(\delta^{-2r})$$

which obeys

$$\tilde{g}_{r-1,t}(x) = -\alpha \tilde{g}_{r,t}(\tilde{g}_{r,t}(x/\alpha))$$

to first order for any choice of d_t, converging to g as $r \to \infty$ for t fixed. That is, (58) is the leading approximation to

$$g_{r,n} = S_{r+n,n} \circ \tilde{g}_{r,r+n} \circ S_{r+n,n}^{-1} \tag{61}$$

which is easily seen to exactly satisfy (35). However, while (58) is *compatible* with the solution (61), (61) is not the *general* solution to (35) containing (58) as its linear approximation. That is, if (35) is stable about the $\tilde{g}_{r,t}$ fixed point, the linear approximation becomes a conjugacy transformation upon $\tilde{g}_{r,t}$, deviations from conjugacy vanishing in the higher order transient.

According to the discussion on p. 680, the functions $\tilde{g}_{r,t}$ all converge to g as $r \to \infty$, but differ in the small-r regime: only $d_t = -1$ (for the properly normalized h) will lead to $\tilde{g}_{0,t}(0) = 0$. Equation (61) is correct for large n; setting $r = 0$, it reads

$$g_{0,n}(x) = S_{n,n} \circ g_{0,n} \circ S_{n,n}^{-1}$$

Since $S_{t,n}$ is connected to the identity,

$$g_{0,n}(0) = 0 \Rightarrow \tilde{g}_{0,n}(0) = 0$$

However, $g_{0,n}(0)$ *must* vanish for all n:

$$g_{0,n}(0) = (-\alpha)^n g^{(2^n)}(\lambda_n, 0) = 0$$

by the recursion-equation definition of λ_n. But, if $\tilde{g}_{0,t}(0) = 0$ for all t, then $d_t = -1$ for all t. Thus, the recursion-defined values of λ_r determine $c_t = \delta^{-t} d_t = -\delta^{-t}$, which, when entered in (58), yields

$$g_{r,n} = g + \sum_\rho c_{r+n}^\rho (-\alpha)^{n(1-\rho)} (g^\rho - x^\rho g') - \delta^{-r}\left(h + \sum_\rho c_{r+n}^\rho (-\alpha)^{n(1-\rho)} (\rho g^{\rho-1} h - x^\rho h')\right) \tag{62}$$

so that the potentially divergent δ^n terms have been stabilized. Moreover, for n large, all terms decay with powers of $-\alpha$ save for $\rho = 1$:

$$g_{r,n} \underset{n\to\infty}{\sim} g + c^1_{r+n}(g - xg') - \delta^{-r}(h + c^1_{r+n}(h - xh'))$$

or, with $\mu_t \equiv 1 + c_t^1$,

$$g_{r,n} \sim \mu_t(g - \delta^{-r}h)(x/\mu_t)$$

or

$$g_{r,n} \sim \mu_{r+n} g_r(x/\mu_{r+n}) \tag{63}$$

a magnification of g_r. Thus, our conjecture implies the local stability of the g_r fixed point of (35). [Conversely, the one parameter λ could be adjusted to cancel the potentially growing δ mode; had $\mathscr{L}$ possessed several growing eigenvalues, it is difficult to see how this cancellation could be arranged. Also, although the conjugacy generators produce convergence at rates $(-\alpha)^{1-\rho}$, we can see from (63) how $\mu_t \to \mu_\infty$ can produce a different convergence scheme for α_n.] We have not investigated any higher order stability questions, and apart from some approximation schemes and computational methods which we shall discuss in the next section, have nothing further to say about the ingredients of a nonlocal proof.

5. APPROXIMATIONS AND METHODS OF SOLUTION

All infinite attracters are locally determined by the hierarchy (8),

$$g_{r-1}(x) = -\alpha g_r(g_r(x/\alpha))$$

As previously described, (8) is solved by first computing g and α through (9),

$$g(x) = -\alpha g(g(x/\alpha))$$

with g_r for asymptotic r given by (32),

$$g_r \sim g - \delta^{-r} h$$

where h and δ are obtained through (28),

$$-\alpha[h(g(x/\alpha)) + g'(g(x/\alpha))h(x/\alpha)] = \delta h(x), \qquad \delta > 1$$

To any desired accuracy, an r_0 is chosen such that (32) provides g_r for all $r \geqslant r_0$, and g_r for $r < r_0$ determined from g_{r_0} through (8). In particular $h(0)$ is fixed through the requirement that $g_0(0) = 0$.

We now seek an approximate equation for g_r for a fixed r that bypasses the above asymptotic ansatz. The virtue of such an equation is that it must define a recursively stable scheme for obtaining g_r. As we shall see, the approximate formula of I shall appear to linear approximation.

By (8),

$$g_1(x) = -\alpha g_2(g_2(x/\alpha)) \tag{64}$$

Relation (32) provides g_r up to first order in δ^{-r}. Since $\delta \to \infty$ as $z \to \infty$, for large enough z

$$g_1 \simeq g - \delta^{-1}h \qquad \text{and} \qquad g_2 \simeq g - \delta^{-2}h \tag{65}$$

will be arbitrarily accurate. Accordingly,

$$g_2 - g_1 \simeq \delta^{-1}(1 - \delta^{-1})h$$

or

$$g_2 \simeq g_1 + \delta^{-1}(1 - \delta^{-1})h \tag{66}$$

Substituting (66) in (64), we find

$$\begin{aligned} g_1(x) = {} & -\alpha\{g_1(g_1(x/\alpha)) + (1 - \delta^{-1})\delta^{-1}[h(g_1(x/\alpha)) \\ & + g_1'(g_1(x/\alpha))h(x/\alpha)] + O(\delta^{-2})\} \end{aligned}$$

or, by (65),

$$\begin{aligned} g_1(x) = {} & -\alpha g_1(g_1(x/\alpha)) + (1 - \delta^{-1})(-\alpha\delta^{-1})[h(g(x/\alpha)) \\ & + g'(g(x/\alpha))h(x/\alpha)] + O(\delta^{-2}) \end{aligned}$$

which, by (28), is

$$g_1(x) = -\alpha g_1(g_1(x/\alpha)) + (1 - \delta^{-1})h(x) + O(\delta^{-2}) \tag{67}$$

Thus, to leading order in δ^{-1},

$$g_0(x) = -\alpha g_1(g_1(x/\alpha)) \simeq g_1(x) - (1 - \delta^{-1})h(x) = g(x) - h(x) \tag{68}$$

That is, as $z \to \infty$, the asymptotic form (32) becomes arbitrarily accurate for all $r \geqslant 0$. Since $g_0(0) = 0$, (68) produces

$$h(0) \simeq g(0) = 1$$

in this limit, a

$$g_1(0) \simeq 1 - \delta^{-1} \tag{69}$$

[To appreciate this estimate, for $z = 2$, $g_1(0) \doteq 0.733$, in comparison with $1 - \delta^{-1} \doteq 0.786$.] Defining

$$g^*(x) \equiv (1 - \delta^{-1})^{-1} g_1((1 - \delta^{-1})x)$$

and

$$h^*(x) \equiv h((1 - \delta^{-1})x)$$

we can write (67) as

$$g^*(x) = -\alpha g^*(g^*(x/\alpha)) + h^*(x) + O(\delta^{-2}) \tag{70}$$

Since

$$-\alpha[h(g_1(x/\alpha)) + g_1'(g_1(x/\alpha))h(x/\alpha)] = \delta h(x) + O(\delta^{-1})$$

we also have

$$h^*(x) = -(\alpha/\delta)[h^*(g^*(x/\alpha)) + g^{*\prime}(g^*(x/\alpha))h^*(x/\alpha)] + O(\delta^{-2}) \quad (71)$$

Equations (70) and (71) are exactly the approximate equations of I, since

$$h^*(0) = 1 \Rightarrow g^*(0) = 1 \qquad \text{by (70)}$$

while the definition of g_1 implies that

$$g^*(1) = 0$$

Since (70) and (71) constitute equations for g_1, their natural recursion forms ($g^*, h^* \rightarrow g_n^*, h_n^*$ on the right-hand sides and g_{n+1}^*, h_{n+1}^* on the left-hand sides) accomplish the recursion

$$g_{1,n+1} = f(g_{1,n})$$

which is stable.

We now exhibit a computational technique for solving (9) based on the observation about Eq. (16). The recursion form of (9) given by (12),

$$\tilde{g}_{n+1}(x) = -\alpha_n \tilde{g}_n(\tilde{g}_n(x/\alpha_n))$$
$$\tilde{g}_n(0) \equiv 1 \qquad \text{for all } n \Rightarrow \alpha_n = -[\tilde{g}_n(1)]^{-1}$$

must be convergent to g if $\tilde{g}_0$ is appropriately chosen. Thus, if f is any function of our class, there is a value λ_∞ such that

$$x_{n+1} = \lambda_\infty f(x_n)$$

will determine the infinite attractor bifurcated from the 2-cycle, in which case Eq. (15),

$$\tilde{g}_0(x) = f(\lambda_\infty x)$$

will lead to convergence. However, the strong instability of (12) requires that λ_∞ be known to very high precision in order that its high iterates will be accurate approximates of g. As a rough estimate, $\tilde{g}_0(x)$ should approximate g, and so (16),

$$f(\lambda_\infty) = \tilde{g}_0(1) \simeq g(1) = -\alpha^{-1}$$

provides an estimate for λ_∞. It is elementary to obtain better estimates. Clearly,

$$\alpha_0 = -[\tilde{g}_0(1)]^{-1} = -[f(\lambda_\infty)]^{-1}$$

and so

$$\tilde{g}_1(x) = [f(\lambda_\infty)]^{-1} f\{\lambda_\infty f(\lambda_\infty x f(\lambda_\infty))\} \quad (72)$$

But now, $\tilde{g}_1(x)$ is a better estimate of g and so

$$\tilde{g}_1(1) \simeq -\alpha^{-1}$$

will provide a better estimate of λ_∞. Accordingly, with α known accurately, we could, for any f of our class, determine high-accuracy estimates of $\lambda_\infty(f)$. With α unknown, we could seek to collaterally determine it with λ_∞ by setting

$$\tilde{g}_1(1) \simeq \tilde{g}_0(1) \tag{73}$$

which by (16) and (72) provides an equation purely for λ_∞. Evidently, by successively setting $\tilde{g}_{n+1}(1) \simeq \tilde{g}_n(1)$ more accurate estimations are obtained. We now show that this can be turned into a highly convergent scheme for λ_∞. It is immediate to see that (12) can be "solved" as

$$\tilde{g}_n(x) = (-1)^n \alpha_{n-1}\alpha_{n-2} \cdots \alpha_0 \tilde{g}_0^{(2^n)}(x/\alpha_{n-1} \cdots \alpha_0)$$

so that

$$\tilde{g}_n(0) = (-1)^n \alpha_{n-1} \cdots \alpha_0 \tilde{g}_0^{(2^n)}(0) \tag{74}$$

Since $\alpha_n \to \alpha$ if $\tilde{g}_0(x) = f(\lambda_\infty x)$ for the exact λ_∞,

$$\tilde{g}_0^{(2^n)}(0) \sim (-\alpha)^{-n} \tag{75}$$

Also, by (74)

$$\alpha_n = -\tilde{g}_0^{(2^n)}(0)/\tilde{g}_0^{(2^{n+1})}(0) \tag{76}$$

so that, with the definition

$$\xi_n \equiv \tilde{g}_0^{(2^{n-1})}(0)\tilde{g}_0^{(2^{n+1})}(0) - [\tilde{g}_0^{(2^n)}(0)]^2 \tag{77}$$

one has

$$\frac{\xi_{n+1}}{\xi_n} = \frac{1}{\alpha_{n+1}\alpha_n} \frac{\alpha_{n+1} - \alpha_n}{\alpha_n - \alpha_{n-1}} \equiv \frac{1}{\alpha_{n+1}\alpha_n} \rho_n \tag{78}$$

with ρ_n the α convergence rate. With

$$\tilde{g}_0(x) = f(ax) \tag{79}$$

and a chosen exactly at $\lambda_\infty(f)$, $\alpha_n \to \alpha$ and so $\rho_n < 1$. Since $\alpha > 1$ for all $z > 1$ ($\alpha \to \infty$ as $z \to 1$ and $\alpha \to 1$ as $z \to \infty$), ξ_n converges to zero α^2 times faster than $\alpha_n \to \alpha$. Accordingly, if one sets $\xi_n = 0$ for each n, an equation for a results (whose solution is a λ_∞ of f) that is exact to linear order in the error of the estimation. (For $f = 1 - 2x^2$, $\xi_n = 0$ yields λ_∞ for the 2-cycle fundamental to $2n$ significant figures and α_n to n significant figures.) Had we wanted λ_∞ for a 3-cycle, $\tilde{g}_0(x) = f^{(3)}(\lambda_\infty x)$ provides the starting $\tilde{g}$, and similarly *all* λ_∞ of a chosen f are rapidly determined (with of course the same α resulting). Once λ_∞ is determined, the iterates of $f(\lambda_\infty x)$ converge toward g until the error in λ_∞ is sufficiently magnified to cause divergence. (A 25-significant-figure estimate of λ_∞ provides a g obeying (9) to one part in 10^{-14} on [0, 1]. As shall follow, one can do significantly better far more quickly.)

We now consider a Newton's-method scheme of solution of (9) which shall lead into deeper considerations of the spectral problem of $\mathscr{L}$, and simple hand computations of λ_r's to several significant figures.

Regarding $g(x)$ on a compact interval—say [0, 1]—as a matrix of its values at N points x_i *together* with an interpolation scheme (of at least zth order to protect a zth-order g), (9) evaluated at the N points x_i becomes a set of N coupled nonlinear equations for the N quantities $g(x_i)$. Accordingly, one can perform an N-dimensional Newton's-method recursion to obtain the $g(x_i)$ from an initial estimate. However, high-precision estimates of g require a high-order interpolation scheme upon a large-N matrix, leading to an inaccurate inversion. Schematically, one writes

$$\bar{g}(x) = g(x) + \delta g(x), \qquad \bar{\alpha} = \alpha + \delta\alpha \tag{80}$$

where $\bar{g}$ and $\bar{\alpha}$ satisfy (9) and g and α serve as an approximate solution. We insert (80) in (9) and expand about the approximate values to first order in δg and $\delta\alpha$. By setting

$$g(0) \equiv \bar{g}(0) = 1$$

in the approximate solution, we have $\delta g(0) = 0$, which determines $\delta\alpha$ in terms of the $\delta g(x)$'s, so that we have a linear equation for $\delta g(x)$ alone. Expressions like $\delta g(g(x/\alpha))$ appear: the equation is evaluated at each x_i and $\delta g(g(x_i/\alpha))$ is expressed through the interpolation procedure in terms of linear combinations of $\delta g(x_i)$. The equations are then inverted to obtain $\delta g(x_i)$ and the procedure iterated. Convergence is slow and precision-limited.

However, the matrix of $g(x_i)$ and interpolation scheme simply constitute a certain parametrization of $g(x)$. Accordingly, one can perform the method with far simpler parametrizations. In particular, setting

$$g_N(x) = 1 + \sum_{i=1}^{N} G_i x^{2i}, \qquad \delta g_N(x) = \sum_{i=1}^{N} \delta G_i x^{2i}$$

and evaluating the linear approximate equation at N points (say $x_m = m/N$) produces an N-dimensional linear system again to be inverted. {In fact, for $z = 2$, precision limitations occurred for $N = 14$, determining α and g consistent with (9) to within 10^{-20} on [0, 1] in 10 sec of CDC 6600 time.} The solution obtained (the G_i) of course provides a very rapidly computable g for any further usage.

With α and g determined, we now face the determination of δ and h from the solution of (28),

$$\mathscr{L}[\psi] = -\alpha[\psi(g(x/\alpha)) + g'(g(x/\alpha)\psi(-x/\alpha)] = \lambda\psi(x)$$

In light of the previous discussion, this is an infinite-dimensional linear eigenvalue problem, which we shall study in a finite-dimensional approximation. That is, set

$$\psi(x) = \sum_{n=0}^{N-1} \psi_n x^{zn} \qquad (\psi_0 \equiv 1) \tag{81}$$

so that (28) evaluated at N points x_i becomes

$$\sum_{n=0}^{N-1} \{\lambda x_i^{zn} + \alpha[g(x_i/\alpha)]^{zn} + \alpha g'(g(x_i/\alpha))(-x_i/\alpha)^{zn}\}\psi_n = 0 \tag{82}$$

or

$$(\lambda X_{in} - \tilde{L}_{in})\psi_n = 0 \qquad \text{(summation convention)} \tag{83}$$

with (83) defining from (82) the $N \times N$ matrices X and $\tilde{L}$. The matrix X is invertible, and so

$$(X^{-1}\tilde{L})_{in}\psi_n \equiv L_{in}\psi_n = \lambda\psi_n \tag{84}$$

Accordingly, the eigenvalues λ are determined from

$$\det(\lambda X - \tilde{L}) = 0 \tag{85}$$

producing N eigenvalues in the N-dimensional approximating space.

The computational results are highly interesting. Starting with $N = 1$ and $x_1 = 0$ there simply results

$$\lambda = -\alpha - \alpha g'(1) = \alpha^z - \alpha$$

which is an approximate formula for δ with $h(1) = h(0) = 1$, asymptotically accurate as $z \to \infty$. Setting $N = 2$ with $x_1 = 0$ and $x_2 = 1$ results in a larger eigenvalue more nearly δ and a smaller one quite close to $\lambda = 1$, with corresponding eigenvectors approximating h and $\psi_1 = g - xg'$. Increasing N and evaluating at equally spaced points in [0, 1] produces more accurate determinations of δ and various $(-\alpha)^{1-\rho}$, $\rho = 1, 2, \ldots$: *δ is the solitary eigenvalue of L greater than* 1, at least in the two cases we studied, $z = 2$ and $z = 4$. {At $N = 14$ we obtain δ and h to 20 places consistent with (28) on [0, 1] and agreeing to the 14 places of our best recursion data.} That is, the spectrum of $\mathscr{L}$ restricted to these discrete linear systems is comprised of the conjugacy eigenvalues smaller than or equal to 1 in absolute value, plus a solitary larger one equal to δ. The eigenvalues are always *nondegenerate*, so that *L is complete* despite its nonsymmetric form. That is, defining L^*, the adjoint of L (the transpose in the present context), and adjoint eigenvectors

$$L^*\psi_\lambda{}^* \equiv \psi_\lambda{}^* L = \lambda\psi_\lambda{}^* \tag{86}$$

normalized to

$$(\psi_{\lambda'}, \psi_\lambda) \equiv \sum_n \psi^*_{\lambda',n}\psi_{\lambda,n} = \delta_{\lambda\lambda'} \tag{87}$$

L can be spectrally decomposed:

$$L_{mn} = \sum_\lambda \lambda\psi_{\lambda,m}\psi^*_{\lambda,n} \tag{88}$$

or symbolically,

$$L = \sum_\lambda \lambda\psi_\lambda\psi_\lambda{}^*$$

so that

$$L^p = \sum_\lambda \lambda^p\psi_\lambda\psi_\lambda{}^*$$

The condition to be met for $L^p\phi$ to not contain any eigenvector $\psi_{\bar\lambda}$ is then

$$(\psi_{\bar\lambda}{}^*, \phi) = 0 \tag{89}$$

We now consider universality in the light of this framework. With $\tilde{g}_0(x) = f(\lambda_\infty x)$,

$$\tilde{g}_0^{(n)}(x) = \lambda_\infty^{-1}(\lambda_\infty f)^{(n)}(\lambda_\infty x)$$

or

$$(\lambda_\infty f)^{(n)}(x) = \lambda_\infty \tilde{g}_0^{(n)}(x/\lambda_\infty) \tag{90}$$

which is simply a magnification by λ_∞ of $\tilde{g}_0^{(n)}(x)$. Defining the recursion

$$\tilde{g}_{n+1}(x) = -\alpha\tilde{g}_n(\tilde{g}_n(x/\alpha)), \qquad \tilde{g}_0(x) = \lambda_\infty f(x) \tag{91}$$

will cause $\tilde{g}_n \to g$, where

$$\tilde{g}_n(x) = (-\alpha)^n \tilde{g}_0^{(2^n)}(x/\alpha^n)$$

Defining $\eta_n \equiv \tilde{g}_n - g$, then

$$\eta_{n+1} = \mathscr{L}[\eta_n]$$

in linear approximation. Expanding η_n along eigenvectors of $\mathscr{L}$, we have by the orthogonality (77)

$$\eta_n(x) = \sum_\lambda \lambda^n(\psi_\lambda{}^*, \eta_0)\psi_\lambda(x) \tag{92}$$

Since $\eta_n \to 0$, the solitary growing mode corresponding to $\lambda = \delta$ must be unexcited. Calling h^* the adjoint eigenvector of eigenvalue δ, this means

$$(h^*, \eta_0) = 0 \tag{93}$$

However,

$$\eta_0 = \tilde{g}_0 - g = \lambda_\infty f - g$$

so that (93) becomes

$$\lambda_\infty(f) = (h^*, g)/(h^*, f) \tag{94}$$

That is, the λ_∞ of the 2-cycle receives the interpretation as the unique value of λ to extinguish the diverging mode of $\mathscr{L}$. [For a fundamental cycle of order s, $\tilde{g}_0$ must be $(\lambda_\infty f)^{(s)}$, so that λ_∞ is no longer multiplicative, although (93) will still provide an equation to determine λ_∞.] For example, for $z = 2$ and $N = 2$

$$g \simeq 1 + g_1 x^2, \qquad h^* \simeq 1 + h_1{}^* x^2, \qquad f \simeq 1 + f_1 x^2$$

and (94) becomes

$$\lambda_\infty(f) \simeq (1 + g_1 h_1{}^*)/(1 + f_1 h_1{}^*) \tag{95}$$

To be a good estimate, η_0 must be in the linear domain, so that f should be "nice." Thus $f = 2x^2$ determines through (95) a 0.1% estimate of λ_∞. Provided f is nice, once g_n and $h_n{}^*$ are determined, (95) allows for 5-sec estimates of $\lambda_\infty(f)$.

In view of the computer spectral evidence, h^* is the unique eigenvector to all conjugacy-generator eigenvectors. This is important in the application of (94): iterates of $\lambda_\infty f$ converge not to g, but to $\mu g(x/\mu)$. In writing $\eta_0 = \lambda_\infty f - g$, we never specified the normalization of g. Indeed, it is irrelevant: h^* is computed from the g normalized to $g(0) = 1$. Since h^* is orthogonal to all conjugacy generators of g,

$$(h^*, \mu g(x/\mu)) = (h^*, g)$$

for all μ (in linear approximation) and so (94) is correct for g with fixed normalization. Moreover, the conjugacy problem of g is solved: if

$$f = \psi \circ g \circ \psi^{-1}$$

for some ψ connected to the identity, then it must follow that

$$(h^*, f) = (h^*, g)$$

and conversely. (Clearly our spectral conjecture is quite strong.) This leads to another method of estimating λ_∞:

$$(\lambda_\infty f - g, h^*) = 0 \Rightarrow \lambda_\infty f \sim g \qquad \text{(conjugacy)}$$

Thus, should λ_∞ satisfy a *necessary* condition for conjugacy, $\lambda_\infty f$ must be conjugate to g. The condition is elementary: if

$$g(x^*) = x^* \qquad \text{and} \qquad \lambda f(\xi_\lambda{}^*) = \xi_\lambda{}^*$$

then

$$g \sim \lambda_\infty f \Rightarrow g'(x^*) = \lambda_\infty f(\xi^*_{\lambda_\infty})$$

But $g'(x^*)$ is a fixed value for fixed z, and so upon calculation of the fixed point of λf an estimate of λ_∞ is had:

$$\lambda_\infty(f) = g'(x^*)/f'(\xi^*_{\lambda_\infty})$$

Again for $f = 1 - 2x^2$, $f = \sin \pi x$, $f = x - x^3$, and other "nice" f's, a 0.1% estimate is obtained for λ_∞. "Nice" here means that f is "close" to conjugate to g.

We now extend these ideas to the $g_{r,n}$ recursion to provide another proof that

$$\lambda_r \sim \lambda_\infty - \mu\delta^{-r}$$

with μ now determined by the same simple kind of estimates (and to equal precision) as was λ_∞. Moreover, we shall demonstrate how the convergence rate of α_n is computable and equal to δ. Repeating Eq. (62),

$$g_{r,n} = g + \sum_\rho c^\rho_{r+n}(-\alpha)^{n(1-\rho)}(g^\rho - x^\rho g')$$
$$- \delta^{-r}\left(h + \sum_\rho c^\rho_{r+n}(-\alpha)^{n(1-\rho)}(\rho g^{\rho-1}h - x^\rho h')\right)$$

it is clear that the $g_{r,n}$ for large n are fixed by determining the $c_t{}^\rho$ from initial data. Thus, for $n = 0$

$$g_{r,0} = \lambda_r f(x) = g + \sum_\rho c_r{}^\rho(g^\rho - x^\rho g') - \delta^{-r}\left(h + \sum_\rho c_r{}^\rho(\rho g^{\rho-1}h - x^\rho h')\right) \tag{96}$$

Recall that

$$\psi_\rho \equiv g^\rho - x^\rho g'$$

is the eigenvector of $\mathscr{L}$ corresponding to $\lambda = (-\alpha)^{1-\rho}$ and so orthogonal to h^*. Defining

$$\rho g^{\rho-1}h - x^\rho h' \equiv h_\rho \tag{97}$$

and projecting (96) on h^*, we have

$$\lambda_r(h^*, f) = (h^*, g) - \delta^{-r}\left(1 + \sum_\rho c_r{}^\rho(h^*, h_\rho)\right) \tag{98}$$

or

$$\lambda_r = \frac{(h^*, g)}{(h^*, f)} - \frac{1 + \sum_\rho c_r{}^\rho(h^*, h_\rho)}{(h^*, f)}\,\delta^{-r} \equiv \lambda_\infty - \mu_r\delta^{-r}$$

Thus,

$$\lambda_\infty = (h^*, g)/(h^*, f)$$

as before and

$$\mu_r/\lambda_\infty = \left[1 + \sum_\rho c_r{}^\rho(h^*, h_\rho)\right]\Big/(h^*, g) \tag{99}$$

Projecting next upon $\psi_{\bar{\rho}}^*$, we have

$$\lambda_r(\psi_{\bar{\rho}}^*, f) = (\psi_{\bar{\rho}}^*, g) + c_r^{\bar{\rho}} - \delta^{-r} \sum_{\rho} c_r^{\rho}(\psi_{\bar{\rho}}^*, h_\rho) \tag{100}$$

In particular, as $r \to \infty$,

$$\lambda_\infty(\psi_{\bar{\rho}}^*, f) = (\psi_{\bar{\rho}}^*, g) + c_\infty^{\bar{\rho}}$$

so that

$$\lim_{r \to \infty} c_r^{\rho} = c_\infty^{\bar{\rho}} = \lambda_\infty(\psi_{\bar{\rho}}^*, f) - (\psi_{\bar{\rho}}^*, g) \tag{101}$$

exists and is finite to meet initial data. Accordingly, for large r we have

$$\lambda_r \sim \lambda_\infty - \mu\delta^{-r}$$

with

$$\frac{\mu}{\lambda_\infty} = \frac{1 + \sum_\rho c_\infty^{\rho}(h^*, h_\rho)}{(h^*, g)}$$

$$= \frac{1 - \sum_\rho (\psi_\rho^*, g)(h^*, h_\rho) + \lambda_\infty \sum_\rho (\psi_\rho^*, f)(h^*, h_\rho)}{(h^*, g)} \tag{102}$$

Accordingly, μ/λ_∞ is also available and easily computed for small N quite accurately.

We next obtain a sum rule for c_r^{ρ}. Setting $x = 0$ in (96), we have

$$\lambda_r = 1 + \sum_\rho c_r^{\rho} - \delta^{-r} h(0)\left(1 + \sum_\rho c_r^{\rho}\right) \tag{103}$$

which as $r \to \infty$ becomes

$$\lambda_\infty = 1 + \sum_\rho c_\infty^{\rho} \tag{104}$$

Since

$$\sum_\rho \rho c_r^{\rho} < \infty$$

by (103),

$$c_r^{\rho} < 1/\rho^{2+\epsilon}$$

for large ρ. Accordingly, truncation of the ρ sum allows high-accuracy estimates. Setting $N = 2$ so that only $\lambda = \delta$ and $\lambda = 1$ contribute, one has the rough result

$$\lambda_\infty \simeq 1 + c_\infty^{1} \tag{105}$$

By (62),

$$g_{r,n} \underset{n\to\infty}{\sim} g + c^1_{r+n}(g - xg') - \delta^{-r}(h + c^1_{r+n}(h - xh'))$$

$$\simeq (1 + c^1_{r+n})(g - \delta^{-r}h)(x/1 + c^1_{r+n}) \tag{106}$$

or,

$$g_{r,n} \underset{n\to\infty}{\longrightarrow} (1 + c_\infty{}^1)g_r(x/1 + c_\infty{}^1) \tag{107}$$

With the rough estimate (105), this reads

$$g_{r,n} \underset{n\to\infty}{\longrightarrow} \lambda_\infty g_r(x/\lambda_\infty)$$

Also, assuming a rapid n approach, and setting $r \to \infty$, we have

$$g_{\infty,n} \simeq \lambda_\infty g(x/\lambda_\infty)$$

so that $g_{\infty,n} \simeq g_{\infty,0}$ is the estimate

$$\lambda_\infty f(x) \simeq \lambda_\infty g(x/\lambda_\infty)$$

or

$$g(x) \simeq f(\lambda_\infty x)$$

Thus we realize that all our approximation schemes produce estimates of the same accuracy. Next, Eqs. (98) and (100) for $N = 2$ are

$$\lambda_r(h^*, f) \simeq (h^*, g) - \delta^{-r}(1 + c_r{}^1(h^*, h_1))$$

and

$$\lambda_r(\psi_1{}^*, f) \simeq (\psi_1{}^*, g) + c_r{}^1(1 - \delta^{-r}(\psi_1{}^*, h_1))$$

The ratio of these equations produces

$$c_r{}^1 - c_\infty{}^1 \propto \delta^{-r} \tag{108}$$

Together with (106), we then have

$$g_{r,n} - g_{r,\infty} \sim \delta^{-n}$$

in fixed r, providing the mechanism for $\alpha_n \to \alpha$ at the rate δ. We are unsure as to why $\alpha_n \to \alpha$ at a rate $\delta' \neq \delta$ for $z > 2$, especially since the spectrum of L for $z = 4$ possesses δ as the unique growing eigenvalue. Presumably, higher order transients can here decay at a rate below that of the "aymptotic" features discussed here. But for this one defect, the above techniques explain to good accuracy every detail of all our recursion data.

6. AFTERWORD

The preceding parts of this paper were contained in a preprint first circulated in November 1976. This paper is incomplete insofar as the uniqueness of an appropriate solution to (9) as well as the basic spectral conjecture remain unproven. Failing to publish it immediately (because it was not self-contained), I allowed it to hover in a limbo while I anticipated some measure of success at a proof, foreshadowing its content in the final section of its predecessor.[(1)]

Early in 1979, I was informed that an effort by Collet *et al.*[(2),2] has succeeded in this task. These authors have proven existence and uniqueness of the appropriate solution to the fundamental equation (9) and verified the spectral conjecture of $\mathscr{L}$. (This demonstration is, so far, restricted to $z = 1 + \epsilon$ with ϵ small.) Accordingly, the theory presented here is now well-founded, although no extension beyond the local stability of the fixed point is expected in the immediate future.

At this time, I should like to mention another effort. In the special case $z = 1 + \epsilon$, ϵ small, it is easy to approximately solve (9) and the spectral problem of $\mathscr{L}$ since α^{-1} is perturbatively small. This result first appeared in a work by Derrida *et al.*[(4)] (DGP). In this and another interesting paper by these authors,[(5)] the work of Metropolis *et al.*[(6)] (MSS) has been significantly elaborated upon through the discovery of an "internal symmetry" of the MSS sequences which allows organizations of these sequences in manners approaching λ_∞ from *above* rather than from below along the harmonics. I will here briefly explore the connection of one aspect of their work with the present work.

There exists a unique fundamental 4-cycle above the λ_∞ of the 2-cycle. Related to the pattern of this 4-cycle by the operation of DGP is a fundamental 8-cycle, below the 4-cycle and closest to λ_∞. Similarly, for each n there is a fundamental 2^n-cycle below the 2^{n-1}-cycle and closest to and above λ_∞. Denoting the parameter value of these cycles that are superstable by $\hat{\lambda}_n$, we have

$$\hat{\lambda}_2 > \hat{\lambda}_3 > \cdots > \hat{\lambda}_n > \cdots > \lambda_\infty \tag{109}$$

DGP observe that

$$\hat{\lambda}_n - \lambda_\infty \propto \delta^{-n}$$

with the same δ as for the harmonics of the 2-cycle. It is easy to see why this can be so. For the harmonics, the functions g_r were constructed, with

$$g_r \sim g - \delta^{-r} h$$

In this form, the coefficient of h is *negative*. Indeed, this is required for the harmonics to guarantee that $g_0(0) = 0$. However, the term in h is perturbative about the fixed point g, so that nothing in the local analysis requires this negative coefficient. Indeed, for an appropriate *positive* coefficient the

[2] See Ref. 3 for a preview.

phenomena described above are explained, with the g's constructed determining the elements of these cycles.

To see how these phenomena are described, write

$$\lambda f = \lambda_\infty f + (\lambda - \lambda_\infty) f \equiv G_0 + (\lambda - \lambda_\infty) H_0 \tag{110}$$

Iterating 2^n times, and keeping terms to order $\lambda - \lambda_\infty$, we obtain

$$(\lambda f)^{2^n} = G_n + (\lambda - \lambda_\infty) H_n + O((\lambda - \lambda_\infty)^2) \tag{111}$$

where

$$G_n = (\lambda_\infty f)^{2^n}$$

Defining

$$\begin{aligned} (-\alpha)^n (\lambda f)^{2^n} (x/(-\alpha)^n) &\equiv f_n \\ (-\alpha)^n G_n(x/(-\alpha)^n) &\equiv g_n \\ (-\alpha)^n H_n(x/(-\alpha)^n) &\equiv h_n \end{aligned} \tag{112}$$

we can write (111) as

$$f_n(x) = g_n(x) + (x - \lambda_\infty) h_n(x) \tag{113}$$

where

$$\begin{aligned} h_{n+1}(x) &= -\alpha[h_n(g_n(-x/\alpha)) + g_n'(g_n(-x/\alpha)) h_n(-x/\alpha)] \\ &\equiv \mathscr{L}_n[h_n(x)] \end{aligned}$$

and

$$h_n = \mathscr{L}_{n-1} \mathscr{L}_{n-2} \cdots \mathscr{L}_0 f \tag{114}$$

By the definition of λ_∞ and g,

$$g_n \to g, \qquad \mathscr{L}_n \to \mathscr{L} \qquad \text{as } n \to \infty$$

Accordingly, (114) becomes

$$h_n \sim c(f) \delta^n h$$

and (113) reads

$$f_n \sim g + c(f)(\lambda - \lambda_\infty) \delta^n h \tag{115}$$

Equation (115) is approximately correct so long as $(\lambda - \lambda_\infty)^n$ is small, which is the case when

$$|\lambda - \lambda_\infty| \delta^n \leqslant \text{small constant}$$

With λ_n chosen as usual to determine the superstable 2^n-cycle harmonic of the 2-cycle,

$$f_n \to g_0$$

Since $g_0(0) = 0$ and $g(0)$, $h(0) \neq 0$ evidently

$$(\lambda_n - \lambda_\infty) \delta^n \sim 1$$

again establishing δ as the λ convergence rate. If an n-independent *finite* condition on the f_n can more generally be maintained, δ will again be the convergence rate and the corresponding limit of the f_n will be given by (115).

Accordingly, consider determining λ_n by the condition that

$$(\lambda_{nf})^{2^n}(\xi_n) = \xi_n$$
$$D(\lambda_{nf})^{2^n}(\xi_n) = \mu \qquad \text{(independent of } n)$$

where ξ_n is the fixed point closest to $x = 0$. By (112) these conditions transcribe to

$$f_n(x_n) = x_n, \qquad f_n'(x_n) = \mu, \qquad x_n = (-\alpha)^n \xi_n$$

As $n \to \infty, f_n \to f_\mu, x_n \to x_\mu$, by (115)

$$x_\mu \sim g(x_\mu) + c(f)(\lambda_n - \lambda_\infty)\delta^n h(x_\mu) \tag{116}$$
$$\mu \sim g'(x_\mu) + c(f)(\lambda_n - \lambda_\infty)\delta^n h'(x_\mu) \tag{117}$$

Denoting the fixed point of g by $\hat{x}$, and the slope of g at $\hat{x}$ by $\hat{\mu}$, then from (116) and (117) we immediately obtain the approximation

$$\lambda_n \sim \lambda_\infty + \delta^{-n}(\hat{\mu} - \mu)/c|h'(\hat{x})| \tag{118}$$

for $\mu \simeq \hat{\mu}$.

Thus, so long as $\mu \neq \hat{\mu}$, the corresponding λ_n converge to λ_∞ at the rate δ. (At $\mu = \hat{\mu}$, $\lambda_n \to \lambda_\infty$ faster than geometric at the rate δ.) Also by (118), the coefficient of h in f_μ, by (115), changes sign at $\mu = \hat{\mu}$: for $|\mu| < |\hat{\mu}|$, f_μ is a g_r or its continuous analog (for example, at bifurcation values) as described at the end of Section 3; for $|\mu| > |\hat{\mu}|$, $\lambda_n \to \lambda_\infty$ from *above*, and evidently the harmonics are not under consideration. For example, with $\hat{\lambda}_n$ of (109)

$$f_n = (-\alpha)^n(\hat{\lambda}_{n+1}f)^{2^n}(x/(-\alpha)^n)$$

corresponds to a limiting value of $|\mu| > |\hat{\mu}|$ and $\hat{\lambda}_n \to \lambda_\infty$ from above at rate δ, as was to be demonstrated. Accordingly, the fixed point g is the "organizing center" for all attractors with $\lambda \to \lambda_\infty$ whether from above or below λ_∞.

As a final comment, it is perhaps worthy to point out the resemblance of the theory presented to the renormalization-group notions of Wilson.[7] Essentially, the function g_r determine elements of infinitely bifurcated attractors at various levels of magnification, belying a self-similarity of their distribution; this structure is precisely reproduced through the operations of composition and rescaling $\mathscr{T}$, resulting in the next lower g_r. The function g itself is the fixed point of $\mathscr{T}$, while the g_r lie on the one-dimensional unstable manifold through g along h; δ and h indeed were determined by linearizing $\mathscr{T}$ about g. More generally, applied to any f, $\mathscr{T}$ can be viewed as a renormalization-group transformation with self-similarity (critical behavior) determined by the fixed point g. Viewing the parameter λ as temperature, λ_∞ is the critical point and δ emerges as a critical exponent. More intuitively, an analog of Kadanoff's block-spin notion is also available. Thus, consider the superstable 2^n cycles starting at $n = 0$, for which there is a single point at $x = 0$. For $n = 1$, this point is split into one at $x = 0$ again, and another point x_1 to the right. For $n = 2$, $x = 0$ again splits, with x_2 nearest to $x = 0$ and to the left, while x_1 splits into a more closely spaced pair with centroid roughly at x_1. By the definition of α, $x_1 \simeq -\alpha x_2$. As n increases, each point splits into a pair with the element nearest to $x = 0$ located $-\alpha$ times nearer to $x = 0$ than its predecessor. It is thus clear that if each closely spaced pair is replaced by a point at its centroid (viewing at lower resolution), then the same set of points about $x = 0$ is reproduced, but with all distances $-\alpha$ times larger. Accordingly, spin-blocking has here the analog of functional composition, while the following volume rescaling is here, rather than a geometrical factor of 2, now a dynamically determined factor of α. In this way, the theory presented in this work may be viewed as an instance arising mathematically of the renormalization-group notions of statistical mechanics.

APPENDIX

We include here some numerical results, useful for normal ($z = 2$) recursive calculations.

A1. $g(x) = 1 + \sum_{i=1}^{7} g_i x^{2i}$, determining g to ten significant figures as [0, 1]:

$$
\begin{aligned}
g_1 &= 1.527632997 \\
g_2 &= 1.048151943 \times 10^{-1} \\
g_3 &= 2.670567349 \times 10^{-2} \\
g_4 &= -3.527413864 \times 10^{-3} \\
g_5 &= 8.158191343 \times 10^{-5} \\
g_6 &= 2.536842339 \times 10^{-5} \\
g_7 &= -2.687772769 \times 10^{-6} \\
&\Rightarrow -g'(1) = \alpha = 2.502907876
\end{aligned}
$$

A2. From the above, $g(x^*) = x^*$ for $x^* = 0.5493052461$ and $g'(x^*) = -1.601191328$, which is required for the estimate

$$\lambda_\infty(f) \simeq g'(x^*)/f'(\xi^*)$$

where ξ^* satisfies

$$\lambda_\infty f(\xi^*) = \xi^*$$

For example, with $f = x(1 - x)$,

$$\xi^* = 1 - \lambda_\infty^{-1} \qquad \text{and} \qquad \lambda_\infty f'(\xi^*) = -\lambda_\infty + 2$$

so that

$$-\lambda_\infty + 2 \simeq g'(x^*)$$

or $\lambda_\infty \simeq 3.60119$, to be compared with the correct result $\lambda_\infty = 3.56995$.

A3. In order for g_r to be computed, one needs $h(x)$ normalized to $h(0) = 1$ together with the correct $h(0)$ to ensure that $g_0(0) = 0$. Regarding $r = 6$ as asymptotic, we have

$$h(0) = 1.318707$$

and a parametrization of similar accuracy to Section A1 is

$$h(x) = h(0)\left(1 + \sum_{i=1}^{6} h_i x^{2i}\right)$$

with

$$\begin{aligned} h_1 &= -3.256513712 \times 10^{-1} \\ h_2 &= -5.055393508 \times 10^{-2} \\ h_3 &= 1.455982806 \times 10^{-2} \\ h_4 &= -8.810422078 \times 10^{-4} \\ h_5 &= -1.062170276 \times 10^{-4} \\ h_6 &= 1.983988805 \times 10^{-5} \end{aligned}$$

Iterating (8), g_1 or g_0 is obtained for estimates of the locations of elements of a highly bifurcated cycle near $x = 0$. Observe that since

$$g_{r-s}(x) = (-\alpha)^s g_r^{(2^s)}(x/(-\alpha)^s)$$

with $s = r = 6$,

$$g_0(x) = (-\alpha)^6 g_6^{(2^6)}(x/\alpha^6) \simeq (-\alpha)^6 (g - \delta^{-6} h)^{(2^6)}(x/\alpha^6)$$

so that g and h restricted to $[0, 1]$ provide g_0 or $[0, \alpha^6]$, thereby determining many elements near $x = 0$.

A4. Solving the eigenvalue problem of L for $N = 2$, we have

$$\delta \simeq 4.6736, \qquad \lambda_1 \simeq 0.9880$$

to be compared with

$$\delta = 4.6692, \qquad \lambda_1 = 1.0000$$

The corresponding eigenvectors and adjoint eigenvectors (unnormalized) are $\left[\psi = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \psi(x) = a + bx^2\right]$

$$h = \begin{pmatrix} 1 \\ -0.3644 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} 1 \\ 1.1082 \end{pmatrix}$$

$$h^* = \begin{pmatrix} 1 \\ -0.9024 \end{pmatrix}, \qquad \psi_1{}^* = \begin{pmatrix} 1 \\ 2.7444 \end{pmatrix}$$

Writing g as

$$g \simeq \begin{pmatrix} 1 \\ g_1 + \frac{1}{3}g_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1.4927 \end{pmatrix}$$

it is trivial to estimate $\lambda_\infty \simeq (h^*, g)/(h^*, f)$. For example, with $f(x) = 1 - 2x^2$, $f = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\lambda_\infty(f) \simeq 0.8368$. Also, $h_1 = \begin{pmatrix} 1 \\ 0.3644 \end{pmatrix}$ and $(h^*, h_1) \simeq 0.5051$ (properly normalized), so that by (102)

$$\frac{\mu}{\lambda_\infty} \simeq \frac{1 - (\psi_1{}^*, g)(h^*, h_1) + \lambda_\infty(\psi_1{}^*, f)(h^*, h_1)}{(h^*, g)} \simeq 0.6851$$

or $\mu \simeq 0.5733$, to be compared with $\mu(f) = 0.5981$. While it is true that $N = 3$ significantly improves this result, this is already quite accurate and trivial to obtain.

ACKNOWLEDGMENTS

The first exact equation, Eq. (9) of the text, together with the scheme of solution incorporating Eq. (91), was obtained by Predrag Cvitanović in discussion and collaboration with the author. This result proved to be seminal in the construction of the theory presented. The author has profited from discussion with R. Menikoff, and thanks his colleagues, especially D. Campbell and F. Cooper, for critical interest.

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